Received January 8, 1767.

III. *A general Inveſtigation of the Nature of the Curve, formed by the Shadow of a prolate Spheroid, upon a Plane ſtanding at right Angles to the Axis of the Shadow; in a Letter to the Royal Society, by Mr.* George Witchell, *F. R. S.*

Gentlemen,

Read Jan. 15, 1767.

I Beg leave to lay before the Royal Society the following inveſtigation of an irregularity in the duration of the eclipſes of Jupiter's ſatellites, occaſioned by the figure of his body.

It has been known for a long time, that Jupiter's body was not truly ſpherical, but a prolate ſpheroid, and that in a much greater degree than any of the other planets; but notwithſtanding this, it was never ſuſpected that it would affect the durations of the eclipſes of the ſatellite, till Dr. Bevis firſt thought of it, in the latter end of the ſummer 1761.

The Doctor, being at that time indiſpoſed, recommended the ſubject to my conſideration; and, in conſequence of his requeſt, I not long after preſented him with a ſolution of the problem, being in ſubſtance the ſame with this, as far as propoſition V. a copy of which he ſoon after tranſmitted to that excellent mathematician the late M. Clairaut.

In March 1763, M. de la Lande, an eminent French aſtronomer, being here, Dr. Bevis ſhewed him my paper; this occaſioned a new article in the *Conn. des Mouv. Celeſt.* 1765, p. 177, under the title, *Inégalité dans les demi-durées des éclipſes des ſatellites de Jupiter, cauſée par l'applatiſſement de Jupiter:* in which he mentions this circumſtance in the following words; "M. le docteur Bevis me fit voir à Londres, au mois "de Mars dernier, une ſolution rigoureuſe & al-"gebraïque de ce probleme, qui conſiſte à trouver la "courbe qui réſulte de la ſection de l'ombre d'un "ſpheroïde à une diſtance quelconque."

In this ſtate it remained ever ſince; for though the Doctor, and ſome other gentlemen, to whom I ſhewed it, frequently urged me to lay it before the Royal Society; I always declined it, till I ſhould have time to make ſome farther additions to it.

A few months ſince, M. Bailly, a French gentleman, publiſhed at Paris an elaborate treatiſe upon the theory of Jupiter's ſatellites; in which he has been pleaſed to give the honour of this diſcovery intirely to M. de la Lande, without the leaſt mention of Dr. Bevis. I then thought it incumbent on me to do juſtice to the Doctor, by immediately finiſhing my paper in the beſt manner I was able, and preſenting it to the Royal Society.

I ſhall be extremely glad, if this rude eſſay ſhould excite ſome more able perſon to treat the ſubject in the manner it deſerves; for though, I believe, my ſolution will not be deficient in point of truth, I am not vain enough to think it may not be performed in a more elegant manner. I have the honour to be,

Gentlemen,

Your moſt obedient

humble ſervant,

Fleet-Street,
Jan. 7, 1767.

George Witchell.

LEMMA.

If any ſpheroid is cut by a plane, in any direction whatever (excepting that which is perpendicular to its axis), the figure of the ſection will be an ellipſis. This is demonſtrated in Simpſon's Fluxions, Vol. II. p. 456.

PROPOSITION I.

Tab. III. fig. 1. Let the ſphere BEGK be cut through its center by the planes BGK, BPD, B*o*D, BOD, EAK, and LPH; it is required to determine the inclination of the planes LPH, BOD, and alſo the inclination of the right lines AC, BC, which is meaſured by the arc AB; there being given the angles of inclination EBF, FB*a*, together with the arc BF: the angles AFB, EAL, being right angles, and the inclination of the required plane BOD, but little exceeding that of the given plane B*o*D.

Let

Philos. Trans. Vol. LVII. TAB. III. p. 30.

Fig. 1.

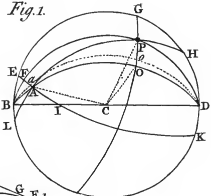
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Fig 2.

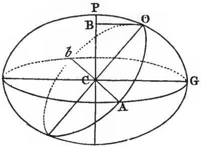

Fig. 3.

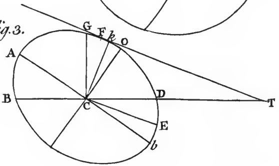
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Fig. 4.

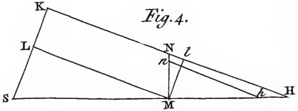

Fig. 5.

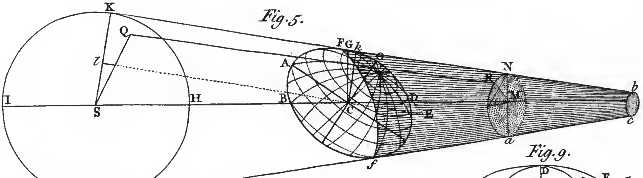
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Fig. 6.

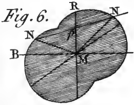

Fig. 7.

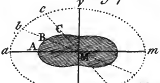

Fig. 8.

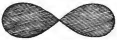

Fig. 9.

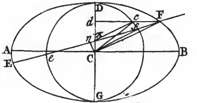
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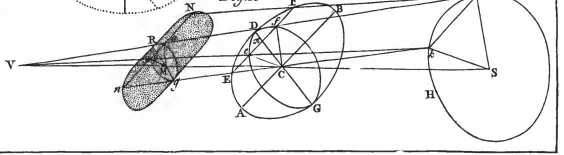
Fig. 10.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Let the ſine of EBF$=a$, its coſine$=a'$, the tangent of BF$=\beta$, its coſine$=b'$, the ſine of FB$a=p$, its co-ſine$=p'$, the ſine of aBA$=z$, the ſine of AB$=$Z, its coſine$=\acute{Z}$, the ſine of PAO $=\zeta$, and radius $=1$; then we ſhall have the ſine of ABF $=$ the ſine of $\overline{\text{FB}a+a\text{BA}}=p+p'z$, and its coſine $=p'-pz$: Therefore by trigonometry we ſhall have in the right angled ſpherical triangle ABF, as Rad. (1) : coſine BF (b') :: ſine ABF ($p+p'z$: coſine BAF$=$ſine of LAB, or its equal PAO; therefore $\zeta=b'\times\overline{p+p'z}=$ the ſine of the required inclination of the planes LPH, BOD. In like manner in the ſame triangle it will be as rad. (1) : cotan. BF $(\frac{1}{\beta})$ :: coſine ABF ($p'-pz$) : cotan. BA $=\frac{\acute{Z}}{Z}$;

hence $Z=\frac{\beta}{\sqrt{\beta^2+\overline{p'-pz}|^2}}$, and $\acute{Z}=\frac{p'-pz}{\sqrt{\beta^2+\overline{p'-pz}|^2}}$, which are the ſine and coſine of the required arc AB.

COROLLARY I.

If inſtead of a ſphere we now ſuppoſe BEGK repreſents a prolate ſpheroid, whoſe axis is CP; the figures of the ſections LPH, BOD, &c. inſtead of circles, will become ellipſes (by the lemma); but it is evident that the inclinations of thoſe planes to each other, and likewiſe the inclination of the right lines AC, BC, or the angle ACB, will remain unaltered.

COROLLARY II.

If BEGK repreſents any primary planet revolving about the ſun in an orbit whoſe plane co-incides with the plane BCD, it is manifeſt that BCD will be its ecliptic, making the angle of obliquity BIE with its equator EAK (whoſe pole is P); and if B be the place of the ſun in this ecliptic, at any given time, the arc BI will be the diſtance of the ſun from the neareſt equinoctial point I; and the arc BF his declination at the ſame time.

COROLLARY III.

If the plane POG, which paſſes through P, the pole of the ſpheroid, be perpendicular to the plane LPH, it will alſo be perpendicular to any other plane BOD, which paſſes through A, the interſection of the equatorial plane EAK, with the plane LPH; therefore the angle ACO being a right angle, it is evident that AC will be the ſemi-tranſverſe, and CO the ſemi-conjugate axis of the elliptic ſection BOD.

COROLLARY IV.

Hence it appears, that the tranſverſe axis of any elliptic ſection BOD, made by a plane paſſing through the center of the ſpheroid, will always be equal to the equatorial diameter of the ſpheroid, but the conjugate axis will be longer or ſhorter, according as the inclination of the planes LPH, BOD, is more or leſs.

PROPOSITION II.

FIG. 2. To find the length of the ſemi-conjugate axis CO, of the elliptic ſection AO*b*, formed by a plane cutting the given prolate ſpheroid POG through its center C, and making the angle PCO with the axis CP.

Let the ſine of the angle PCO $= \zeta$, CG $= t$, CP $= c$, CO $= \varkappa$, (radius being unity); draw PO perpendicular to CP: then in the right angled plane triangle BCO, we have as rad. (1) : CO ($\varkappa$) :: ſine PCO (ζ) : BO ($\varkappa\zeta$); and rad. (1) : CO ($\varkappa$) :: coſine PCO ($\sqrt{1-\zeta^2}$) : BC ($\varkappa\sqrt{1-\zeta^2}$); but from the nature of the ellipſis we have $\frac{c^2}{t^2} \times \overline{t^2 - \varkappa^2\zeta^2}$ $= \overline{BC}|^2 = \varkappa^2 - \varkappa^2\zeta^2$; therefore $\varkappa^2 = \frac{t^2c^2}{t^2 - \overline{t^2 - c^2} \times \zeta^2}$, or putting $t^2 - c^2 = f^2$, and $t^2 - \varkappa^2 = \varphi^2$, we have $\varkappa^2 = \frac{t^2c^2}{t^2 - f^2\zeta^2}$, and $\varphi^2 = \frac{t^2 \times \overline{1 - \zeta^2}}{\frac{t^2}{f^2} - \zeta^2}$.

PROPOSITION III.

Fig. 3. Let BOD be an ellipſis, whoſe tranſverſe diameter Ab makes the angle ACB, with the right line BCD, and let TkG be a tangent to the ellipſis, in the point F, making the angle GTC with the right line BCD: It is required to find the length of the normal Ck, drawn from the center of the ellipſis, to the tangent TG.

From C, the center of the ellipſis, let CE be drawn parallel to the tangent TG, meeting the ellipſis in the point E; and CG perpendicular to the line BCD, meeting the tangent in the point G: Put the ſine of ACB $= Z$, its coſine $= \overset{\prime}{Z}$, the ſine of TGC $= V$, its coſine $= \overset{\prime}{V}$ (radius being unity) AC $= t$, CO $= \varkappa$, and $t^2 - \varkappa^2 = \varphi^2$; then will the ſine of OCE (= the ſine of OCD + DCE) be expreſſed by $ZV + ZV'$, and by the laſt propoſition we ſhall find CE $= \frac{t\varkappa}{\sqrt{t^2 - \varphi^2 \times \overline{Z'V + ZV'}^{2}}}$; but (by conics) $CE \times Ck = CO \times CA$, whence we ſhall obtain $Ck = \sqrt{t^2 - \varphi^2 \times \overline{Z'V + ZV'}^{2}}$.

PROPOSITION IV.

Fig. 4. In the two ſimilar right angled plane triangles HKS, HMN, right angled at K, and M, there is given the right lines KS and MS, to find the acute angles, ſuppoſing the given angle hnM to be nearly equal to the required angle HNM. Put $MS = \Delta$, $KS = r$, $MN = \upsilon$, the ſine of the given angle $hn\,M = q$, its coſine $= q'$, the ſine of HNM $= V$, its coſine $= V'$, the ſine of $\overline{HNM - hn\,M} = x$, and radius $= 1$. Let ML be drawn parallel to HK, and Ml parallel to SK: then in the right angled plane triangles NMl, SML, we have as rad. (1) : MN (υ) : : ſine HNM (V) : Ml (υV), and as rad. (1) : MS (Δ) : ſin. LMS (V') : LS ($\Delta V'$); but $Ml + LS = KS$; therefore $\upsilon V + \Delta V' = r$, and by the foregoing notation $V = q + q'x$, and $V' = q' - qx$; therefore theſe values of V and V' being wrote in the above equation we ſhall find $x = \frac{q'\Delta + q\upsilon - r}{q\Delta - q'\upsilon}$, and from thence $V = \frac{\Delta - q'r}{q\Delta - q'\upsilon}$, and $V' = \frac{qr - \upsilon}{q\Delta - q'\upsilon}$.

PROPOSITION V.

Fig. 5. If the opake prolate ſpheroid BPOD, given in ſpecies and poſition, be oppoſed to the given luminous ſphere HKQI at the given diſtance CS, forming the ſhadow F*fbc*: It is propoſed to determine the figure of the ſection *a*RN made by a plane, cutting the ſhadow perpendicularly to its axis at the given diſtance MS.

Let the required curve *a*RN be conceived to be generated by the extremity R, of the variable right line MR, revolving about the given point M as a center, the line MR being always perpendicular to the axis of the ſhadow MS: Let the right line RQ be a tangent to the ſphere HKQI in the point Q, and in the ſame plane with the right lines RM, MS, it will then repreſent one of the rays of light, which conſtitute the conical ſuperficies of the ſhadow, and, therefore, by the laws of optics, will be a tangent to the ſpheroid alſo; now when the generating point R has arrived at N, the ray RQ (being ſuppoſed to revolve with it) will co-incide with the tangent NK, touching the ſphere in K, and the ſpheroid in F: Join K, S, and the angles NMS, and NKS, will be right angles; let the ſpheroid be ſuppoſed to be cut, by the quadrangular plane NMSK, forming thereby the elliptic ſection BOD, draw C*k* perpendicular, and C*l* parallel to NK; put $CA = t$, $MC = \delta$, $CS = d$, $MS = \Delta$, $SK = r$, $MN = \upsilon$, $CO = \varkappa$, the ſine of *b*NM $= V$, its coſine $= \overset{\prime}{V}$, the ſine of ACB $= Z$, its coſine $= Z$, and radius $= 1$:

Then in the right angled plane triangle ClS, it will be as rad. (1) : CS (d) :: ſine SCl ($\acute{V}$) : Sl ($d\acute{V}$), and conſequently Ck (= KS — Sl) = $r - d\acute{V}$; but by prop. III. Ck = $\sqrt{t^2 - \phi^2 \times \overline{\acute{Z}V + Z\acute{V}}^2}$, whence we ſhall have $r - d\acute{V} = \sqrt{t^2 - \phi^2 \times \overline{\acute{Z}V + Z\acute{V}}^2}$: Now by propoſition I. we ſhall find $\zeta = b' \times \overline{p + p'z}$, $Z = \frac{\beta}{\sqrt{\beta^2 + \overline{p' - pz}^2}}$, and $\acute{Z} = \frac{p' - pz}{\sqrt{\beta^2 + \overline{p' - pz}^2}}$; by prop. II. $\varkappa^2 = \frac{t^2 c^2}{t^2 - f^2 \zeta^2}$, and $\phi^2 = \frac{t^2 \times \overline{1 - \zeta^2}}{\frac{t^2}{f^2} - \zeta^2}$; laſtly by prop. IV. $V = \frac{\Delta - q'r}{q\Delta - q'v}$, and $\acute{V} = \frac{qr - v}{q\Delta - q'v}$, which values being ſubſtituted in the above equation will exhibit the nature of the required curve aRN, in terms of z and v.

SCHOLIUM.

If the ſphere HKQI repreſents the ſun, and the ſpheroid BPOD one of the primary planets, it will appear, from the preceding reaſoning, that the figure of the ſection of its ſhadow received upon a plane, which is perpendicular to its axis, will not be a circle (except when the axis of the planet produced paſſes through the ſun's center) but a curve of the oval kind, whoſe ſpecies will be known from the foregoing equation.

If the ſphere HKQI had been regarded as a ſpheroid in the above ſolution, it is eaſy to ſee that the foregoing proceſs would have determined the nature of the required curve; but the figure of the ſun is ſo nearly ſpherical, that it was not thought neceſſary to embarraſs the ſolution with that conſideration.

Hence the duration of an eclipſe of a given ſatelles may be determined in the following manner: Let B R C (fig. 6.) be the ſection of the ſhadow, through which the ſatelles paſſes, N p N the path of the ſatelles, making the given angle N p M, with the circle of latitude R p M; B M C a part of the primary's orbit produced, and M p the given latitude of the ſatelles at the time of the ſyzygia; the circle of latitude R p M is repreſented in fig. 1. by the primitive circle B E G D, and the angle R M N, by the ſpherical angle E B A; therefore the ſine of R M N = the ſine of E B A = the ſine of $\overline{EBF + FBa + aBA}$ $= ap' + a'p + \overline{a'p' - ap} \times z$, and its coſine $= a'p' - ap - \overline{ap' + a'p} \times z$; which for the ſake of brevity may be expreſſed by y, and y'; then putting $Mp = n$, $MN = v$, the ſine of $MpN = m$, its coſine $= m'$, and radius $= 1$; we ſhall have the ſine of M N P expreſt by $my' + m'y$; and therefore we ſhall have in the plane triangle MpN, as the ſin. MNp $(my' + my) : Mp\ (n) :: $ ſine $MpN\ (m) : MN\ (v)$; hence $v = \frac{m.n}{my' + ym'}$; from which, and the equation of the curve (determined above) $\frac{vy}{m} = pN$, and conſequently, the duration of the eclipſe will become known.

In prop. I. the ſine of the angle ABF is expreſſed by $p + p'z$, and its coſine by $p' - pz$, inſtead of their true values $pz' + p'z$, and $p'z' - pz$; this was done to render the following concluſions more ſimple than they otherwiſe would have been; and as the angle *a* B A is, by hypotheſis, but ſmall, its coſine will approach ſo near to the radius, as not to occaſion any ſenſible error in the reſult; and the ſame may be obſerved with regard to what is advanced in prop. IV.

It remains now to apply, what has been inveſtigated above, to the eclipſes of Jupiter's ſatellites, and to examine whether the prolateneſs of his figure will have any ſenſible effect upon their durations; and this is become the more neceſſary, as that celebrated aſtronomer M. de la Lande (who candidly acknowledges, that he was excited to turn his thoughts upon this ſubject, from a curſory view of this paper, which was ſhewn him by Dr. Bevis *) does not ſeem to have conſidered the queſtion, with that degree of attention which I think it demands.

But before this can be done with exactneſs, it will be neceſſary to have the inclination of Jupiter's axis, with reſpect to his ecliptic, and the place of his equinoxes determined by obſervation, neither of which I believe has yet been done with any degree of certainty; I ſhall, therefore, proceed in this inquiry upon M. de la Lande's hypotheſis, that Jupiter's axis is perpendicular to his orbit; and perhaps this ſuppoſition is not ſo far diſtant from the truth, as to occaſion any material error in the concluſion. It may alſo be remarked, that in the general equation given above, $\grave{V}$ and V expreſs the ſine and coſine of the ſemi-angle of the cone of Jupiter's ſhadow, but this angle can never exceed 3′, and conſequently we may very ſafely uſe the radius inſtead of V wherever it occurs.

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* Vid. Connoiſſ. des Mouv. Celeſt. 1765, p. 177.

By this means the general equation will become $r - d\overset{\prime}{V} = \sqrt{t^2 - \varphi^2}$, or which is the ſame $r - d\overset{\prime}{V} = x$, therefore $\overset{\prime}{V} = \frac{r - x}{d}$; but by prop. IV. $\overset{\prime}{V} = \frac{qr - v}{q\Delta - q'v}$, which, becauſe q is nearly equal to V, and $q'v$ very ſmall with reſpect to $q\Delta$, will become $\overset{\prime}{V} = \frac{r - v}{\Delta}$; therefore $\frac{r - x}{d} = \frac{r - v}{\Delta}$, from which we ſhall find $x = \frac{\Delta x - \delta r}{d}$; and this equation is exactly the ſame with that which would ariſe from conſidering the ſun as a circular, and Jupiter as an elliptic plane, limited by one of his meridians, and always parallel to the diſk of the ſun; which ſuppoſition, the immenſe diſtance of Jupiter from the ſun renders very allowable.

From this equation an eaſy mechanical method may be derived of delineating the curve of the ſhadow, at any given diſtance from Jupiter, for as x denotes any ſemi-diameter of the elliptic ſection of Jupiter's body, it is manifeſt, that the term $\frac{\Delta}{d} \times x$, will expreſs the correſponding ſemi-diameter of a ſimilar ellipſis, whoſe axes are to thoſe of Jupiter in the given ratio of Δ to d, and the term $\frac{\delta r}{d}$ is wholly given: Therefore if *a r m* (fig. 7.) be ſuch an ellipſis, and there be drawn through its center M any number of ſemi-diameters M*a*, M*b*, M*c*, &c. meeting the ellipſis in a, b, c, &c. let aA, bB, cC, &c. be taken each equal to the given term $\frac{\delta r}{d}$, and the points A,B,C, &c. will be in the required curve.

It appears from conſidering the nature of this curve, that it will have two cuſps, one at each extremity of its leſſer axis, which will approach toward each other, according as the diſtance δ is augmented; therefore, if the diſtance of the ſection of the ſhadow, from Jupiter's center, was taken, ſuch that $\delta = \frac{dc}{r-c}$, the leſſer axis of the curve would then vaniſh, and the cuſps meet in the center, and thereby form two diſtinct ſhadows (as repreſented in fig. 8); in conſequence of which, if a ſatelles, revolved at that diſtance, it might ſuffer a double eclipſe, at the ſame conjunction, which remarkable phænomenon may alſo happen, at a leſs diſtance from Jupiter, in ſome circumſtances.

I ſhall now ſhew how the duration of an eclipſe of a given ſatelles may be determined independant of the equation of the curve; and this, perhaps, will be the more acceptable, as it will afford a practical rule, which may be applied, in every poſition of Jupiter's axis, with very little trouble. This may be done by the help of the following propoſition.

PROPOSITION VI.

If a circle $eDfG$ be deſcribed about the conjugate axis GD, of a given ellipſis ADBG, and a right line EF be drawn, making the given angle $F\pi D$, with the conjugate axis, and paſſing through the given point π taken therein, it is propoſed to determine the length of the ſegments Ff, Ee, intercepted between the circumference of the circle, and the perimeter of the ellipſis.

From the point F, draw the right line Fd parallel to the tranſverſe axis AB, meeting the conjugate GD in the point d, and the circle in c; draw the lines CF, Cf, Cc, and let $c\pi$ be joined: Then by conics we ſhall have, as CB : CD :: tang. FπD : tang. $c\pi$D, and in the right lined triangle Cπc, it will be as Cc (CD) : ſin. Cπc :: Cπ : ſin. C$c\pi$, whence the angle cCπ becomes known; but as CD : CB :: tang. cCπ : tang. FCπ; therefore FCπ is known; from which taking away the given angle fCπ, there remains the angle FCf; conſequently all the angles, in the right lined triangle fCF, together with the ſide Cf (CD), are known: we ſhall therefore have, in the right lined triangle, FfC, as ſin. fFC : Cf :: ſin. fCF : fF, one of the required ſegments, and by a ſimilar operation, the other ſegment Ee will be found, whence as ef is given, EF will become known.

COROLLARY I.

The required ſegments Ff, Ee, will be found in the ſame manner, when the given point π is not taken in one of the axes, but any where between; but in that caſe, the point where the line EF interſects the conjugate axis, muſt be firſt determined.

COROLLARY II.

If a perpendicular Cn be let fall from C upon the line EF, the angle πCn will be given, to which, adding $FC\pi$ (found above) the angle FCn will be known; hence we ſhall have the following analogy for determining Fn: As tang. fCn : tang. FCn :: $fn : Fn$.

Now let KkH (fig. 10.) repreſent the diſk of the ſun, and $eDfG$ that of Jupiter, conſidered as a circle, whoſe diameter is equal to his axis DG, draw Npn, the path of the ſatelles, making the given angle NpR, with a right light Rg drawn parallel to the diameter DG, and let ab be the duration of the eclipſe, and V the apex of the ſhadow in this hypotheſis; join Va, Vb, and let the plane aVb be produced, till it meets the ſun's diſk in K and k, it will then interſect the diſk of Jupiter in the line $f\pi e$, and the lines VK, Vk, will alſo touch the circumference of the circle $eDfG$, in the points e and f, draw the line SV, and it will be the axis of the ſhadow, and conſequently will paſs through C and M, the centers of Jupiter, and the ſection of the ſhadow; join aM, bM, fC, eC, and the triangles abM, efC, will be ſimilar to each other, and, therefore, abM being wholly given, feC will likewiſe be known. Let ADBG be the elliptic ſection of Jupiter's body, and produce $e\pi f$ both ways, till it meets the periphery of the ellipſis in the points E and F, draw KF, kE, and produce them till they meet with ab, produced both ways in N and n, then will Nn be the required duration of the eclipſe in the true ſhadow: Now the triangles KfF, KaN, being ſimilar, as are alſo the triangles keE, kbn, and the ſegments Ff, eE, being given by the preceding propoſition, the required ſegments Na, bn, will alſo become known, for they will be to the former ſegments in the given ratio of SM to SC.

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It may be obſerved, that this method is equally applicable, whether the axis of Jupiter is perpendicular to his orbit, or not; for if it is not, we can eaſily find by propoſition I and II. the ſpecies and poſition of that elliptic ſection of Jupiter's body, to which a right line connecting the centers of the ſun and Jupiter is perpendicular; and this being obtained every thing elſe will remain as before.

As it would require more time, than I have to ſpare at preſent, to enter into a particular inquiry concerning the alterations, which this irregularity in the ſhadow will occaſion, in the preſent theory of Jupiter's ſatellites, I ſhall conclude with obſerving, that the errors in the ſemi-durations of their eclipſes, ariſing from this cauſe, may ſometimes amount to $20''$ in the firſt; $50''$ in the ſecond; $2'\ 19''$ in the third; and $11'\ 14''$ in the fourth; which errors will, I believe, be deemed ſufficiently large to merit the attention of aſtronomers.

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